Price expectations model with second-order differential equations

Given the following model:

$$Q_d = a + b\,p + c\,p' + d\,p''$$
$$Q_s = e + f\,p + g\,p' + h\,p''$$

with $a > 0$, $b < 0$, $e < 0$, $f > 0$

$$Q_d = Q_s$$

where $p = p(t)$, $p'(t) = \frac{dp}{dt}$ and $p''(t) = \frac{d^2p}{dt^2}$.

1. Find the differential equation that defines the model.
2. Solve the following problem knowing that $p(0) = 6$ and $p'(0) = 4$:

$$Q_d = 42 - 4p - 4p' + p''$$
$$Q_s = -6 + 8p$$
$$Q_d = Q_s$$

Solution

1. Equate supply and demand:

$$a + bp + cp' + dp'' = e + fp + gp' + hp''$$

Rearrange:

$$p''(d-h) + p'(c-g) + p(b-f) = e - a$$

and divide by $d-h$

$$p'' + p'\frac{(c-g)}{d-h} + p\frac{(b-f)}{d-h} = \frac{e-a}{d-h}$$

2. This model is a particular case where $a = 42$, $b = -4$, $c = -4$, $d = 1$, $e = -6$, $f = 8$, $g = 0$, $h = 0$. Using these values:

$$p'' - 4p' - 12p = -48$$

We propose the homogeneous solution:

$$p_H = e^{rt}$$
$$p'_H = e^{rt}r$$
$$p''_H = e^{rt}r^2$$
$$r^2 - 4r - 12 = 0$$

The roots are $r = -2$ and $r = 6$. Thus, the homogeneous solution is:

$$p_H = C_1e^{-2t} + C_2e^{6t}$$

Now for the particular solution. Since the right-hand side only has a constant, we propose:

$$p_c = A$$

Thus:

$$p'_c = p''_c = 0$$

Substituting:

$$-12A = -48$$
$$A = 4$$
$$p_c = 4$$

Finally, the general solution:

$$p_g = p_H + p_c = C_1e^{-2t} + C_2e^{6t} + 4$$

Using the initial conditions:

$$C_1 + C_2 + 4 = 6$$

And since $p' = -2C_1e^{-2t} + 6C_2e^{6t}$

$$4 = -2C_1 + 6C_2$$

Dividing this equation by 2:

$$2 = -C_1 + 3C_2$$

Adding to the first equation:

$$4C_2 = 4$$

Solving for C_2:

$$C_2 = 1$$

And then finding C_1:

$$C_1 = 1$$

Finally:

$$p_g = e^{-2t} + e^{6t} + 4$$